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Nonautonomous Submersive Second-Order Differential Equations and Lie Symmetries

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We give necessary and sufficient conditions for a nonautonomous second-order differential equation to be submersive. An application to nonautonomous Lagrangian systems is given: the existence of symmetries of the Lagrangian permits us to prove that the Euler–Lagrange vector field is submersive and hence that the motion equations may be simplified. Our results extend to the nonautonomous case the previous ones obtained by Kossowski and Thompson.

1. INTRODUCTION

The purpose of this paper is to characterize submersive nonautonomous second-order differential equations (SODEs) in order to extend the results of Kossowski and Thompson (1991) to the nonautonomous situation. [See also Martínez *et al.* (1993) for the study of separability of SODEs.] A nonautonomous system of second-order differential equations is submersive if there exist local coordinates such that the system contains a subsystem with fewer coordinates. On the other hand, a nonautonomous system of second-order differential equations may be interpreted as a vector field ξ_M on the stable tangent bundle $\mathbb{R} \times TM$ of some manifold M. Then the submersive character may be reinterpreted as the existence of a foliation on M in such a way the vector field ξ_M projects to the local quotients. Since a foliation is defined by a family of local submersions satisfying some compatibility conditions, we can introduce the notion of global submersive nonautonomous SODEs as a vector field on $\mathbb{R} \times TM$ for which there exists a surjective submersion $\rho: M \to N$ and a new nonautonomous SODE ξ_N on

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$\mathbb{R} \times TN$ such that ξ_M projects onto ξ_N. We also characterize the global submersive character of ξ_M.

Nonautonomous SODEs appear in the geometric formulation of nonautonomous Lagrangian mechanics. In fact, if $L: \mathbb{R} \times TM \to \mathbb{R}$ is a regular Lagrangian, then the corresponding Euler–Lagrange vector field ξ_L is a nonautonomous SODE (Cantrijn *et al.*, 1992; de León and Rodrigues, 1988, 1989, 1990). In this paper we establish the relationship between the submersive character of ξ_L and the existence of some Lie subalgebras of Lie symmetries of ξ_L. Since a symmetry of L is also a symmetry of ξ_L, the existence of symmetries of the Lagrangian permits us to simplify the motion equations. We remark that this procedure is different from those of Marsden and Weinstein (1974; Abraham and Marsden, 1978; Marsden, 1992) for the autonomous situation (symplectic reduction) and Albert (1989; Cantrijn *et al.*, 1992; de León and Saralegui, n.d.) for the nonautonomous situation (cosymplectic reduction). A main difference is that by using the actual procedure, we obtain a projected nonautonomous SODE on $\mathbb{R} \times TN$, while applying the cosymplectic reduction procedure, we have that the reduced Hamiltonian vector field is not in general a nonautonomous SODE (Marsden *et al.*, 1990). Another difference is that we can reconstruct the dynamics in a direct way since we can lift the solutions of a projected nonautonomous SODE to the solutions of the submersive nonautonomous SODE by fixing the initial conditions. These differences are shown in the last section by exhibiting a particular example.

We notice that our results extend those of Kossowski and Thompson (1991), which can be recovered from the present ones.

The paper is organized as follows. Section 2 is devoted to a brief background on tangent and stable tangent geometry, second-order differential equations and connections, and nonautonomous Lagrangians systems. In Section 3 submersive nonautonomous SODEs are introduced and a geometric characterization of the submersiveness is given. In Section 4 we study the relationship between the existence of some Lie algebras of Lie symmetries with the submersive character of a nonautonomous SODE. Furthermore, we prove that the existence of a Lie symmetry of a nonautonomous Lagrangian can be used to simplify the motion equations. In Section 5 we will illustrate our method by means of an example. We also apply the cosymplectic reduction to it and analyze the different results.

2. BACKGROUND

2.1. Tangent and Stable Tangent Geometry

Let M be a manifold of dimension m and TM its tangent bundle. Then TM carries a canonical integrable *almost tangent structure* J_M (Grifone, 1974; de León and Rodrigues, 1989). If (q^i, v^i) are induced coordinates in TM, then we have

$$J_M\left(\frac{\partial}{\partial q^i}\right)=\frac{\partial}{\partial v^i}, \qquad J_M\left(\frac{\partial}{\partial v^i}\right)=0$$

Another geometrical ingredient of TM is the *Liouville vector field* C_M, the infinitesimal generator of dilations on TM, and it is locally expressed by $C_M = v^i(\partial/\partial v^i)$. The *evolution space* $J^1(\mathbb{R}, M)$ is the manifold of jets of order one (de León *et al.*, 1992; de León and Rodrigues, 1988, 1989, 1990). It is clear that $J^1(\mathbb{R}, M)$ may be canonically identified with $\mathbb{R} \times TM$ since TM is the submanifold of $J^1(\mathbb{R}, M)$ of 1-jets with fixed source $0 \in \mathbb{R}$. We denote by

$$\tau_M: \quad TM \to M, \qquad \tilde{\tau}_M: \quad \mathbb{R} \times TM \to \mathbb{R} \times M$$

the canonical projections defined by

$$\tau_M(j_0^1\sigma) = \sigma(0), \qquad \tilde{\tau}_M(j_t^1\sigma) = (t, \sigma(t))$$

In local coordinates we have

$$\tau_M(q^i, v^i) = (q^i), \qquad \tilde{\tau}_M(t, q^i, v^i) = (t, q^i)$$

The key geometric structure of the evolution space $J^1(\mathbb{R}, M)$ is the almost stable-tangent structure (almost s-tangent structure, for simplicity). Almost s-tangent structures are the odd-dimensional counterpart of almost tangent structures according to the following definition (Oubiña, 1983).

Definition 2.1. Let V be a $(2m+1)$-dimensional manifold. If there is a triple $(\bar{J}, \tau, T)$, where $\bar{J}$ is a tensor field of type (1, 1), τ is a 1-form, and T a vector field on V such that

(i) $i(T)\tau = 1$
(ii) $\bar{J}^2 = T \otimes \tau$
(iii) $\operatorname{rank} \bar{J} = m + 1$

then we say that V is endowed with an *almost s-tangent structure*. In such a case V is called an *almost s-tangent manifold.*

We define a tensor field $\bar{J}_M$ on $\mathbb{R} \times TM$ by $\bar{J}_M = J_M + (\partial/\partial t) \otimes dt$. In local coordinates we have

$$\bar{J}_M\left(\frac{\partial}{\partial t}\right)=\frac{\partial}{\partial t}, \qquad \bar{J}_M\left(\frac{\partial}{\partial q^i}\right)=\frac{\partial}{\partial v^i}, \qquad \bar{J}_M\left(\frac{\partial}{\partial v^i}\right)=0$$

Thus $(\bar{J}_M, dt, \partial/\partial t)$ is an almost s-tangent structure on $J^1(\mathbb{R}, M)$ (Oubiña, 1983).

An almost s-tangent structure $(\bar{J}, \tau, T)$ is *integrable* if it is locally equivalent to $(\bar{J}, dt, \partial/\partial t)$. One can easily prove that $(\bar{J}, \tau, T)$ is integrable if and only if its Nijenhuis tensor $N_{\bar{J}}$ vanishes and $d\tau = 0$ (Oubiña, 1983; de León *et al.*, 1992).

From now on we shall assume the integrability of $(\bar{J}, \tau, T)$ as a G-structure, i.e., around each point of V there exists a coordinate system (t, q^i, v^i) such that

$$\bar{J}\left(\frac{\partial}{\partial t}\right) = \lambda\left(\frac{\partial}{\partial t}\right), \qquad \bar{J}\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial v^i}, \qquad \bar{J}\left(\frac{\partial}{\partial v^i}\right) = 0, \qquad T = \frac{\partial}{\partial t}, \qquad \tau = dt$$

For the sake of simplicity we assume that $\lambda = 1$ (in the case $\lambda = -1$ we proceed in a similar way).

Let us recall that there is defined a canonical tensor field on $J^1(\mathbb{R}, M)$ given by $\tilde{J}_M = J_M - C_M \otimes dt$. Hence $\tilde{J}_M$ has rank m and satisfies $(\tilde{J}_M)^2 = 0$. Locally,

$$\tilde{J}_M\left(\frac{\partial}{\partial t}\right) = -C_M; \qquad \tilde{J}_M\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial v^i}; \qquad \tilde{J}_M\left(\frac{\partial}{\partial v^i}\right) = 0$$

Proposition 2.1. Let $\rho\colon M \to N$ be a smooth mapping and denote by $T\rho\colon TM \to TN$ the induced tangent mapping. Then we have

(i) $T(\mathrm{Id}_{\mathbb{R}} \times T\rho)C_M = C_N$
(ii) $T(\mathrm{Id}_{\mathbb{R}} \times T\rho)\, \partial/\partial t = \partial/\partial t$
(iii) $T(\mathrm{Id}_{\mathbb{R}} \times T\rho)(J_M Y) = J_N(T(\mathrm{Id}_{\mathbb{R}} \times T\rho)Y)$
(iv) $T(\mathrm{Id}_{\mathbb{R}} \times T\rho)(\tilde{J}_M Y) = \tilde{J}_N(T(\mathrm{Id}_{\mathbb{R}} \times T\rho)Y)$

where Y is tangent vector to $\mathbb{R} \times TM$.

Proof. It follows by a direct computation in local coordinates. ∎

2.2. Lifts of Vector Fields and Distributions

Let X be a vector field on a manifold M. We denote by X^C the *complete lift* of X to TM defined as follows: if Φ_t is the flow generated by X on M, then $T\Phi_t$ is the flow generated by X^C on TM. The *vertical lift* X^V of X to TM is now defined by $X^V = J_M X^C$. Thus, if X is locally written as

$$X = X^i \frac{\partial}{\partial q^i}$$

then we have

$$X^C = X^i \frac{\partial}{\partial q^i} + v^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial v^i}, \qquad X^V = X^i \frac{\partial}{\partial v^i}$$

Next, we shall define the different types of lifts of vector fields and distributions to $\mathbb{R} \times TM$. We denote by (t, q^i, τ, v^i) the induced coordinates on $T(\mathbb{R} \times M)$.

We denote by $\imath_k: \mathbb{R} \times TM \to T(\mathbb{R} \times M)$, $k \in \mathbb{R}$, the canonical injection defined by

$$\imath_k(j_t^1\sigma) = j_0^1\sigma', \qquad \sigma'(s) = (t + ks, \sigma(s))$$

Hence in local coordinates we have

$$\imath_k(t, q^i, v^i) = (t, q^i, k, v^i)$$

Let X be a vector field on $\mathbb{R} \times M$. We set

$$X^v = X^V - d\tau(X^V)\frac{\partial}{\partial \tau}$$

where X^V is the vertical lift of X to $T(\mathbb{R} \times M)$. Then the *vertical lift* X^{V_k} of X to $\mathbb{R} \times TM$ is the restriction of X^v to the submanifold $\imath_k(\mathbb{R} \times TM)$, say

$$X^{V_k} = (X^v)_{|\imath_k(\mathbb{R} \times TM)}$$

If X is locally written as

$$X = A\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i}$$

then we have

$$X^{V_k} = X^i\frac{\partial}{\partial v^i}$$

Thus, we deduce that

$$X^{V_k} = X^{V_{k'}} = X^V, \qquad \forall k, k' \in \mathbb{R}$$

We notice that X^V is precisely the vertical lift of X to TM considered as a vector field on $\mathbb{R} \times TM$.

In a similar way we can define the complete lifts as follows. First we set

$$X^c = X^C - d\tau(X^C)\frac{\partial}{\partial \tau}$$

where X^C is the vertical lift of X to $T(\mathbb{R} \times M)$. Then the *complete lift* X^{C_k} of X to $\mathbb{R} \times TM$ is the restriction of X^c to the submanifold $\imath_k(\mathbb{R} \times TM)$, say

$$X^{C_k} = (X^c)_{|\imath_k(\mathbb{R} \times TM)}$$

Then we have

$$X^{C_k} = A\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i} + k\frac{\partial X^i}{\partial t}\frac{\partial}{\partial v^i} + v^j\frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial v^i}$$

If X is a vector field on M, then we have

$$X^{C_k} = X^{C_{k'}}, \qquad \forall k, k' \in \mathbb{R}$$

We notice that X^{C_0} is precisely the complete lift of X to TM considered as a vector field on $\mathbb{R} \times TM$.

For a vector field X on $\mathbb{R} \times M$ we denote by $X^{(1)}$ its canonical prolongation to the jet manifold $J^1(\mathbb{R}, M)$ (Prince, 1983; Saunders, 1989). If X is locally expressed by

$$X = A\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i}$$

then we have

$$X^{(1)} = A\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i} + \bar{X}^i\frac{\partial}{\partial v^i}$$

where

$$\bar{X}^i = \frac{\partial X^i}{\partial t} - v^i\frac{\partial A}{\partial t} + v^j\left(\frac{\partial X^i}{\partial q^j} - \frac{\partial A}{\partial q^j}v^i\right)$$

Then if $\langle dt, X\rangle = 0$, we obtain $X^{(1)} = X^{C_1}$. Moreover, if X is a vector field on M, then $X^{(1)} = X^{C_0}$.

Let D be a distribution on M. Then D may be lifted to a distribution $\tilde{D}$ on TM as follows. If $\{X_1, \ldots, X_r\}$ is a local basis of D, then $\{X_1^V, \ldots, X_r^V, X_1^C, \ldots, X_r^C\}$ is a local basis of $\tilde{D}$. Hence $\tilde{D}$ has even dimension, say $2r$. A distribution E on TM is called a *tangent* distribution if it is of the form $E = \tilde{D}$ for some distribution D on M (Cantrijn *et al.*, 1986). E is called J_M*-regular* if, moreover, D is a regular distribution on M. Here regular means that D is involutive (hence D defines a foliation on M also denoted by D) and the leaf space M/D determined by the foliation D is a quotient manifold. Of course, D is involutive (regular) iff E is involutive (regular).

A distribution D on M may be considered as a distribution on $\mathbb{R} \times M$. Then the natural lift $\tilde{D}$ is a distribution on $\mathbb{R} \times TM$. In the sequel we shall consider distributions on $\mathbb{R} \times M$ (and on $\mathbb{R} \times TM$) obtained from distributions on M (and on TM).

2.3. Second-Order Differential Equations and Dynamical Connections

We say that a vector field ξ_M on $J^1(\mathbb{R} \times M)$ is a *nonautonomous second-order differential equation* (or a *nonautonomous SODE* for simplicity) if and only if $\tilde{J}_M \xi_M = 0$ and $J_M \xi_M = C_M$. In such a case ξ_M is locally given by

$$\xi_M = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial q^i} + \xi^i_M(t, q, v) \frac{\partial}{\partial v^i}$$

Obviously, ξ_M is a nonautonomous SODE if and only if $\tilde{J}_M \xi_M = (\partial/\partial t) + C_M$.

A curve $\sigma\colon \mathbb{R} \to M$ is called a *solution* of ξ_M if its canonical prolongation $j^1\sigma\colon \mathbb{R} \to \mathbb{R} \times TM$ is an integral curve of ξ_M. Thus, if $\sigma(t) = (q^i(t))$, then σ is a solution of ξ_M if and only if it satisfies the following system of nonautonomous second-order differential equations:

$$\frac{d^2 q^i}{dt^2} = \xi^i_M\left(t, q^j, \frac{dq^i}{dt}\right) \tag{1}$$

A (nonhomogeneous) *connection* (Grifone, 1972) on M (i.e., a connection in the fibration $TM \to M$) is a tensor field Γ of type $(1, 1)$ on TM such that $J_M \Gamma = J_M$ and $\Gamma J_M = -J_M$. This notion can be extended to jet bundles of order one as follows.

A *dynamical connection* (de León and Rodrigues, 1988, 1989, 1990) on $J^1(\mathbb{R}, M)$ is a tensor field Γ of type $(1, 1)$ such that

$$J_M \Gamma = \tilde{J}_M \Gamma = \tilde{J}_M, \qquad \Gamma \tilde{J}_M = -\tilde{J}_M, \qquad \Gamma J_M = -J_M$$

Then the local expressions of Γ are

$$\Gamma\left(\frac{\partial}{\partial t}\right) = -v^i \frac{\partial}{\partial q^i} + \Gamma^i \frac{\partial}{\partial v^i}$$

$$\Gamma\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial q^i} + \Gamma^j_i \frac{\partial}{\partial v^j}$$

$$\Gamma\left(\frac{\partial}{\partial v^i}\right) = -\frac{\partial}{\partial v^i}$$

The operators $l = \Gamma^2$ and $m = \mathrm{Id} - \Gamma^2$ are complementary projection operators of an almost product structure on $J^1(\mathbb{R}, M)$. The local expressions of l and m are

$$l\left(\frac{\partial}{\partial t}\right) = -v^i \frac{\partial}{\partial q^i} - (\Gamma^i + v^j \Gamma^i_j) \frac{\partial}{\partial v^i}$$

$$l\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial q^i}$$

$$l\left(\frac{\partial}{\partial v^i}\right) = \frac{\partial}{\partial v^i}$$

$$m\left(\frac{\partial}{\partial t}\right)=\frac{\partial}{\partial t}+v^i\frac{\partial}{\partial q^i}+(\Gamma^i+v^j\Gamma^i_j)\frac{\partial}{\partial v^i}$$

$$m\left(\frac{\partial}{\partial q^i}\right)=m\left(\frac{\partial}{\partial v^i}\right)=0$$

If we set $\mathbf{L}=l(J^1(\mathbb{R},M))$ and $\mathbf{M}=m(J^1(\mathbb{R},M))$, then they are complementary distributions. We deduce that $\mathbf{L}$ (resp. $\mathbf{M}$) is a $2n$-dimensional (resp. one-dimensional) distribution, locally spanned by $\{\partial/\partial q^i, \partial/\partial v^i\}$ [resp. globally spanned by $(\mathrm{Id}-(\Gamma)^2)(\partial/\partial t)$]. The vector field $\xi_\Gamma=m(\partial/\partial t)$ is locally expressed by

$$\xi_\Gamma=\frac{\partial}{\partial t}+v^i\frac{\partial}{\partial q^i}+(\Gamma^i+v^j\Gamma^i_j)\frac{\partial}{\partial v^i}$$

i.e., it is a nonautonomous SODE, called a *canonical nonautonomous SODE associated to* Γ. We now set $h=(1/2)(\mathrm{Id}+\Gamma)l$, $v=(1/2)(\mathrm{Id}-\Gamma)l$; then h and v are complementary projectors in $\mathbf{L}$. Thus, $\mathbf{H}=h(\mathbf{L})$, $\mathbf{V}=v(\mathbf{L})$ are complementary distributions in $\mathbf{L}$, i.e., $\mathbf{L}=\mathbf{H}\oplus\mathbf{V}$. Moreover, the distribution $\mathbf{V}$ is locally spanned by $\{\partial/\partial v^i\}$ and then it is precisely the vertical distribution of the fibration $\tilde{\tau}_M\colon \mathbb{R}\times TM\to\mathbb{R}\times M$. Hence Γ defines in fact a connection in $\tilde{\tau}_M\colon\mathbb{R}\times TM\to\mathbb{R}\times M$ whose horizontal distribution is $\mathbf{H}\oplus\mathbf{M}$.

Let Γ be a dynamical connection. Then a tangent vector Y on $\mathbb{R}\times TM$ which belongs to $\mathbf{H}$ (resp. $\mathbf{H}'=\mathbf{H}\oplus\mathbf{M}$) will be called a *strong* (resp. *weak*) horizontal tangent vector. If Y is a vector field on $\mathbb{R}\times TM$, then Y will be called a *strong* (resp. *weak*) horizontal vector field if $Y_z\in\mathbf{H}_z$ (resp. $Y_z\in\mathbf{H}'_z$) for any $z\in\mathbb{R}\times TM$. If X is a vector field on $\mathbb{R}\times M$, then there exists a unique vector field $X^{H'}$ on $\mathbb{R}\times TM$ which is weak horizontal and projects onto X. We call $X^{H'}$ the *weak horizontal lift* of X and its $\mathbf{H}$ component, denoted by X^H, is called the *strong horizontal lift* of X to $\mathbb{R}\times TM$. A direct computation in local coordinates shows that

$$\left(\frac{\partial}{\partial t}\right)^{H'}=\frac{\partial}{\partial t}+\left(\Gamma^j+\frac{1}{2}v^i\Gamma^j_i\right)\frac{\partial}{\partial v^j},\qquad \left(\frac{\partial}{\partial q^i}\right)^{H'}=\frac{\partial}{\partial q^i}+\frac{1}{2}\Gamma^j_i\frac{\partial}{\partial v^j}$$

Hence, if X is a vector field on M and $X=X^i(\partial/\partial q^i)$, then we obtain

$$X^{H'}=X^i\frac{\partial}{\partial q^i}+\frac{1}{2}X^i\Gamma^j_i\frac{\partial}{\partial v^j}$$

It is clear that $\Gamma X^{H'}=X^{H'}$, since $\Gamma h=h$.

A curve $\sigma\colon\mathbb{R}\to M$ is a *geodesic* of a dynamical connection Γ if the jet prolongation $j^1\sigma$ of σ is a weak horizontal curve on $\mathbb{R}\times TM$. Then σ is a geodesic of Γ if and only if it satisfies the following system of second-order differential equations:

$$\frac{d^2q^i}{dt^2} = \Gamma^i\left(t, q, \frac{dq}{dt}\right) + \Gamma^i_j\left(t, q, \frac{dq}{dt}\right)\frac{dq^j}{dt} \tag{2}$$

From (1) and (2) we deduce that the geodesics of a dynamical connection Γ are precisely the solutions of its associated nonautonomous SODE.

In de León and Rodrigues (1988, 1989, 1990) it is shown that if ξ_M is a given nonautonomous SODE on $J^1(\mathbb{R}, M)$, then Γ_{ξ_M} defined by $\Gamma_{\xi_M} = -\mathscr{L}_{\xi_M}\tilde{J}_M$ is a dynamical connection on $J^1(\mathbb{R}, M)$ whose associated nonautonomous SODE is precisely ξ_M. If

$$\xi_M = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial q^i} + \xi^i_M(t, q, v)\frac{\partial}{\partial v^i}$$

then we have

$$\begin{aligned}
\Gamma_{\xi_M}\left(\frac{\partial}{\partial t}\right) &= -v^i\frac{\partial}{\partial q^i} - \left(v^j\frac{\partial\xi^i_M}{\partial v^j} - \xi^i_M\right)\frac{\partial}{\partial v^i}\\
\Gamma_{\xi_M}\left(\frac{\partial}{\partial q^i}\right) &= \frac{\partial}{\partial q^i} + \frac{\partial\xi^j_M}{\partial v^i}\frac{\partial}{\partial v^j}\\
\Gamma_{\xi_M}\left(\frac{\partial}{\partial v^i}\right) &= -\frac{\partial}{\partial v^i}
\end{aligned}$$

Let ξ_M be a nonautonomous SODE on $\mathbb{R} \times TM$ and $\Gamma_{\xi_M} = -\mathscr{L}_{\xi_M}\tilde{J}_M$ the associated connection. Since the nonautonomous SODE associated to Γ_{ξ_M} is just ξ_M, then we deduce that the solutions of ξ_M are precisely the geodesics of Γ_{ξ_M}.

From a direct computation in local coordinates we obtain the following result.

Proposition 2.2. We have

1. $\Gamma_{\xi_M}(X^V) = -X^V$
2. $\Gamma_{\xi_M}(X^{C_0}) = -[\xi_M, X^V]$
3. $X^{H'} = X^{C_0} - \frac{1}{2}([\xi_M, X^V] + X^{C_0})$

for any vector field X on M.

2.4. Nonautonomous Lagrangian Systems

Let $L: \mathbb{R} \times TM \cong J^1(\mathbb{R}, M) \to \mathbb{R}$ be a nonautonomous Lagrangian. The *energy function* associated to L is defined by $E_L = C_M L - L$. The *Poincaré–Cartan 1-form* associated to L is defined by

$$\alpha_L = d_{J_M}L - E_L\,dt$$

and then the *Poincaré–Cartan 2-form* associated to L is

$$\Omega_L = -d\alpha_L = -dd_{J_M}L + dE_L \wedge dt$$

In local coordinates we obtain

$$E_L = v^i \frac{\partial L}{\partial v^i} - L$$

$$\alpha_L = \frac{\partial L}{\partial v^i} dq^i - E_L\, dt$$

$$\Omega_L = -\left(\frac{\partial^2 L}{\partial v^i \partial t} + v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - \frac{\partial L}{\partial q^i}\right) dt \wedge dq^i$$
$$+ v^j \frac{\partial^2 L}{\partial v^j \partial v^i} dv^i \wedge dt - \frac{\partial^2 L}{\partial v^i \partial q^j} dq^j \wedge dq^i - \frac{\partial^2 L}{\partial v^i \partial v^j} dv^i \wedge dq^j$$

If L is regular, i.e., the Hessian matrix $(\partial^2 L/\partial v^i \partial v^j)$ is nonsingular, then (Ω_L, dt) is a cosymplectic structure on $J^1(R, Q)$ (de León and Rodrigues, 1988, 1989, 1990; Cantrijn *et al.*, 1992). Consequently, there exists a unique vector field ξ_L on $J^1(\mathbb{R}, M)$ (the *Euler–Lagrange vector field*) such that

$$i_{\xi_L}\Omega_L = 0, \qquad i_{\xi_L} dt = 1 \tag{3}$$

ξ_L is a nonautonomous SODE and its solutions are just the solutions of the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0; \qquad v^i = \frac{dq^i}{dt}; \qquad 1 \le i \le m$$

Let $L: J^1(\mathbb{R}, M) \to \mathbb{R}$ be a regular Lagrangian. Let ξ_L be the Euler–Lagrange vector field. Then there exists a dynamical connection Γ on $J^1(\mathbb{R}, M)$ whose geodesics are solutions of the above motion equations. This connection is given by $\Gamma_L = -\mathscr{L}_{\xi_L}\tilde{J}_M$.

3. SUBMERSIVE NONAUTONOMOUS SECOND-ORDER DIFFERENTIAL EQUATIONS

We shall study the following problem. Suppose that ξ_M is a nonautonomous SODE on $\mathbb{R} \times TM$. We search for the existence of a local coordinate system (x^α, y^a), $1 \le \alpha \le n$, $1 \le a \le m - n$, around each point of M such that the following system of second-order differential equations

$$\frac{d^2 q^i}{dt^2} = \xi^i_M\left(t, q^j, \frac{dq^j}{dt}\right), \qquad 1 \le i \le m \tag{4}$$

may be written as

$$\frac{d^2x^\alpha}{dt^2} = \xi^\alpha_M\left(t, x^\beta, \frac{dx^\beta}{dt}\right)$$
$$\frac{d^2y^a}{dt^2} = \xi^a_M\left(t, x^\beta, y^b, \frac{dx^\beta}{dt}, \frac{dy^b}{dt}\right)$$

In such a case we say that ξ_M is *locally submersive.* Notice that the existence of such coordinates is equivalent to the existence of a foliation on M, or, in other words, to the existence of a family of local submersions $\rho_A: (x^\alpha, y^a) \to (x^\alpha)$ defining this foliation. ξ_M is called *globally submersive* (or simply *submersive*) if there exists a global surjective submersion $\rho: M \to N$ of M onto a manifold N such that

$$T(\mathrm{Id}_{\mathbb{R}} \times T\rho)\xi_M = \xi_N$$

where ξ_N is a nonautonomous SODE on $\mathbb{R} \times TN$.

The purpose of this section is to obtain geometric conditions for ξ_M to be submersive. First we have the following result.

Proposition 3.1. If $\rho: M \to N$ is a surjective submersion and ξ_M is a nonautonomous SODE on $\mathbb{R} \times TM$ which is $(\mathrm{Id}_{\mathbb{R}} \times T\rho)$-related to some vector field ξ_N on $\mathbb{R} \times TN$, then ξ_N is a nonautonomous SODE.

Proof. The result follows directly from the definition of nonautonomous SODE and Proposition 2.1. ■

Now suppose that ξ_M is a submersive nonautonomous SODE on $\mathbb{R} \times TM$. Then there exists a surjective submersion $\rho: M \to N$ and a nonautonomous SODE on $\mathbb{R} \times TN$ such that ξ_M and ξ_N are $(\mathrm{Id}_{\mathbb{R}} \times T\rho)$-related, i.e., we have $T(\mathrm{Id}_{\mathbb{R}} \times T\rho)\xi_M = \xi_N$. We obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{R} \times TM & \xrightarrow{pr_2} & TM & \xrightarrow{\tau_M} & M \\
\Big\downarrow{\scriptstyle \mathrm{Id}_{\mathbb{R}} \times T\rho} & & \Big\downarrow{\scriptstyle T\rho} & & \Big\downarrow{\scriptstyle \rho} \\
\mathbb{R} \times TN & \xrightarrow{pr_2} & TN & \xrightarrow{\tau_N} & N
\end{array}$$

where $pr_2: \mathbb{R} \times TM \to TM$ and $pr_2: \mathbb{R} \times TN \to TN$ are the canonical projections onto the second factor. Suppose that $\dim M = m$ and $\dim N = n$. Then the involutive distribution $D = \mathrm{Ker}\, T\rho$ has dimension $m - n$ and its canonical lift $E = \tilde{D}$ is precisely $E = \mathrm{Ker}\, T(T\rho)$. It is clear that E is a J_M-regular distribution on TM. We denote by the same letter E the induced distribution on $\mathbb{R} \times TM$. Of course, E has dimension $2(m - n)$.

Now, let $\Gamma_{\xi_M} = -\mathscr{L}_{\xi_M}\tilde{J}_M$ be the dynamical connection on $\mathbb{R} \times TM$ determined by ξ_M. From Proposition 2.2 we have

$$\Gamma_{\xi_M} X^V = -X^V$$
$$\Gamma_{\xi_M} X^{C_0} = -[\xi_M, X^V]$$

for any vector field $X \in D$. Since E is locally generated by the vertical and C_0-lifts of vector fields belonging to D and since ξ_M is submersive, we deduce that E is Γ_{ξ_M}-invariant.

Next, since

$$(\mathscr{L}_{\xi_M}\Gamma_{\xi_M})(Z) = [\xi_M, \Gamma_{\xi_M} Z] - \Gamma_{\xi_M}[\xi_M, Z]$$

we obtain by the submersiveness of ξ_M and the Γ_{ξ_M}-invariance of E that E is also $\mathscr{L}_{\xi_M}\Gamma_{\xi_M}$-invariant.

The main result of this section shows that these properties are a geometric characterization of submersive nonautonomous SODEs.

Theorem 3.1. A nonautonomous SODE ξ_M on $\mathbb{R} \times TM$ is submersive to a nonautonomous SODE ξ_N on $\mathbb{R} \times TN$ if and only if there exists a distribution E on $\mathbb{R} \times TM$ which is J_M-regular and Γ_{ξ_M}- and $\mathscr{L}_{\xi_M}\Gamma_{\xi_M}$-invariant.

Proof. We only need to prove the sufficiency. In fact, from the results of Crampin and Thompson (1985) and Thompson and Schwardmann (1991) (also see de León and Rodrigues, 1989) it follows that there exists a commutative diagram as above. We need to show that ξ_M is $(\mathrm{Id}_{\mathbb{R}} \times T\rho)$-projectable. In such a case its projection will be a nonautonomous SODE because of Proposition 2.2. But ξ_M is $(\mathrm{Id}_{\mathbb{R}} \times T\rho)$-projectable if and only if

(i) $[\xi_M, X^V] \in E$
(ii) $[\xi_M, X^{C_0}] \in E$

for any vector field $X \in D$, where $E = \tilde{D}$ and $D = \operatorname{Ker} T\rho$, $E = \operatorname{Ker} T(T\rho)$.

(i) Since $[\xi_M, X^V] = -\Gamma_{\xi_M}(X^{C_0})$, we deduce (i) from the Γ_{ξ_M} invariance of E.

(ii) A direct computation in local coordinates shows that $[\xi_M, X^V] + X^{C_0}$ is vertical. Then we only need to prove

(ii)′ $[\xi_M, X^{H'}] \in E$

for any vector field $X \in D$. Furthermore, from (i) and Proposition 2.2 we deduce that $X^{H'} \in E$ for any $X \in D$. Since $\Gamma_{\xi_M}(X^{H'}) = X^{H'}$, we have

$$(\mathscr{L}_{\xi_M}\Gamma_{\xi_M})(X^{H'}) = [\xi_M, X^{H'}] - \Gamma_{\xi_M}[\xi_M, X^{H'}]$$

But $l([\xi_M, X^{H'}]) = [\xi_M, X^{H'}]$ and hence

$$\frac{1}{2}(\mathscr{L}_{\xi_M}\Gamma_{\xi_M})(X^{H'}) = \frac{1}{2}(\mathrm{Id} - \Gamma_{\xi_M})[\xi_M, X^{H'}] = v([\xi_M, X^{H'}])$$

where $v = \frac{1}{2}(\mathrm{Id} - \Gamma_{\xi_M})l$. Consequently, we obtain that $v([\xi_M, X^{H'}]) \in E$.

On the other hand, we have

$$X^{H'} = \Gamma_{\xi_M}(X^{H'}) = -(\mathscr{L}_{\xi_M}\tilde{J}_M)(X^{H'}) = -[\xi_M, X^V] + \tilde{J}_M[\xi_M, X^{H'}]$$

which implies

$$\tilde{J}_M[\xi_M, X^{H'}] = X^{H'} + [\xi_M, X^V]$$

and thus $\tilde{J}_M[\xi_M, X^{H'}] \in E$. But since $[\xi_M, X^{H'}] \in \mathbf{L}$, we have

$$[\xi_M, X^{H'}] = h[\xi_M, X^{H'}] + v[\xi_M, X^{H'}]$$

Then we only need to prove that $h[\xi_M, X^{H'}] \in E$. To do this, we take a local basis $\{X_\alpha, Y_a\}$ of vector fields on M such that

$$D = \langle Y_a \rangle$$

$$E = \langle Y_a^V, Y_a^{C_0} \rangle$$

Thus, we have

$$[\xi_M, X^{H'}] = \lambda_a Y_a^V + \mu_a Y_a^{C_0} + A_\alpha X_\alpha^V + B_\alpha X_\alpha^{C_0}$$

Since

$$\tilde{J}_M[\xi_M, X^{H'}] = \mu_a Y_a^V + B_\alpha X_\alpha^V \in E$$

we deduce that $B_\alpha = 0$. Hence

$$h[\xi_M, X^{H'}] = \mu_a Y_a^{H'} \in E$$

This ends the proof. ■

Corollary 3.1. Let $E_1, \ldots, E_r$ be r regular tangent distributions on TM such that $E_u \cap (+_{v \neq u} E_v) = 0$. Then the Whitney sums $E_{u_1} \oplus \cdots \oplus E_{u_s}$ are well defined, where $u_1, \ldots, u_s \in \{1, \ldots, r\}$, and $s \leq r$. Suppose that the Whitney sums $\hat{E}_u = \bigoplus_{v \neq u} E_v$ are also regular. If TTM splits as a Whitney sum $TTM = E_1 \oplus \cdots \oplus E_r$, then M is a product manifold, say $M = N_1 \times \cdots \times N_r$ and $\xi_M = (\xi_{N_1} + \cdots + \xi_{N_r})_{|\mathbb{R} \times TN_1 \times \cdots TN_r}$, where ξ_{N_u} is a non-autonomous SODE on $\mathbb{R} \times TN_u$, $1 \leq u \leq r$.

Proof. We remark that if $E_u = \tilde{D}_u$, then the distributions $D_1, \ldots, D_r$ on M verify the same properties as the distributions $E_1, \ldots, E_r$ on TM and we have $\hat{E}_u = \tilde{\hat{D}}_u$, where $\hat{D}_u = \bigoplus_{v \neq u} D_v$. Since $\hat{D}_u$ is also regular we obtain submersions $\rho_u : M \to N_u$, for each u, where $N_u = M/\hat{D}_u$, and non-autonomous SODE's ξ_{N_u} which are $(\mathrm{Id}_{\mathbb{R}} \times T_\rho)$-projections of ξ_M. We define the mapping $\rho : M \to N_1 \times \cdots \times N_r$ by $\rho(x) = (\rho_1(x), \ldots, \rho_r(x))$, $x \in M$. Hence ρ is a diffeomorphism and $T(\mathrm{Id}_{\mathbb{R}} \times T_\rho)(\xi_M) = (\xi_{N_1} + \cdots + \xi_{N_r})_{|\mathbb{R} \times TN_1 \times \cdots TN_r}$. ■

If ξ_M satisfies the hypotheses of Corollary 3.1, we say that ξ_M is *decomposable*. If, in particular, TTM splits as a direct sum of rank 2 subbundles, then ξ_M is called *separable*. In such a case there exist local coordinates around each point of M such that (4) may be written as follows:

$$\frac{d^2q^1}{dt^2} = \xi_M^1\left(t, q^1, \frac{dq^1}{dt}\right)$$
$$\frac{d^2q^m}{dt^2} = \xi_M^m\left(t, q^m, \frac{dq^m}{dt}\right)$$

The autonomous case was extensively studied by Martínez *et al.* (1993).

Remark 3.1. We notice that if in Theorem 3.1 the assumption of regularity is removed, then the nonautonomous SODE ξ_M is only locally submersive.

Remark 3.2. Suppose that ξ_M is a SODE on TM. Then $\tilde{\xi}_M = \partial/\partial t + \xi_M$ is a nonautonomous SODE on $\mathbb{R} \times TM$. We denote by

$$\Gamma_{\xi_M} = -\mathscr{L}_{\xi_M} J_M, \qquad \tilde{\Gamma}_{\tilde{\xi}_M} = -\mathscr{L}_{\tilde{\xi}_M} \tilde{J}_M$$

the connection on TM and the dynamical connection on $\mathbb{R} \times TM$ determined by ξ_M and $\tilde{\xi}_M$, respectively. A direct computation in local coordinates shows that

$$\tilde{\Gamma}_{\tilde{\xi}_M} = \Gamma_{\xi_M} - (\mathscr{L}_{C_M}\xi_M) \oplus dt \tag{5}$$

Now, let E be a distribution on TM. Then from (5) we deduce that E is Γ_{ξ_M}- and $\mathscr{L}_{\xi_M}$-invariant if and only if E is $\tilde{\Gamma}_{\tilde{\xi}_M}$- and $\mathscr{L}_{\tilde{\xi}_M}\tilde{\Gamma}_{\tilde{\xi}_M}$-invariant. Hence we deduce that the main result of Kossowski and Thompson (1991) (Theorem 1.5) may be reobtained from Theorem 3.1.

To end this section we exhibit how we can obtain the solutions of the nonautonomous SODE ξ_M from the solutions of the projected nonautonomous SODE ξ_N. It is clear that the solutions of ξ_M project onto the solutions of ξ_N. Conversely, if $\sigma_N: \mathbb{R} \to N$ is a solution of ξ_N, then we can lift σ_N to a solution of ξ_M, but this lift is not unique. However, if we fix initial data on $\mathbb{R} \times TM$, then there exists a unique lift. Also, if $f: \mathbb{R} \times TN \to \mathbb{R}$ is a first integral of ξ_N, say $\xi_N f = 0$, then its lift $f \circ (\mathrm{Id}_{\mathbb{R}} \times T\rho)$ is a first integral of ξ_M.

4. LIE SYMMETRIES AND NONAUTONOMOUS SECOND-ORDER DIFFERENTIAL EQUATIONS

Let ξ_M be a nonautonomous SODE on $\mathbb{R} \times TM$. A vector field X on $\mathbb{R} \times TM$ such that

$$[X^{(1)}, \xi_M] = -\xi_M(\langle dt, X\rangle)\xi_M$$

will be called a *Lie symmetry* of ξ_M (Prince, 1983, 1985; de León and Marrero, 1993). We restrict ourselves to those Lie symmetries X which are vector fields on M. Such a Lie symmetry will be called an *autonomous Lie symmetry* of ξ_M. In such a case the above condition becomes

$$[X^{(1)}, \xi_M] = [X^{C_0}, \xi_M] = 0$$

Since $[X^{C_0}, Y^{C_0}] = [X, Y]^{C_0}$ for any vector fields X, Y on M, we deduce that the set of autonomous Lie symmetries of ξ_M is a Lie subalgebra of the Lie algebra $\mathscr{X}(\mathbb{R} \times M)$ of vector fields on $\mathbb{R} \times M$.

Let $\mathscr{G}$ be a Lie subalgebra of autonomous Lie symmetries of ξ_M. We know (Cantrijn *et al.*, 1986; Kossowski and Thompson, 1991) that $\mathscr{G}$ determines an involutive distribution $\tilde{\mathscr{G}}$ on TM (and hence on $\mathbb{R} \times TM$) as follows:

$$\tilde{\mathscr{G}} = \{X^V, X^{C_0} | X \in \mathscr{G}\}$$

We shall prove that the existence of some Lie subalgebras of autonomous Lie symmetries of ξ_M implies the submersive character of ξ_M.

In the sequel the horizontal lifts are considered with respect to the dynamical connection $\Gamma_{\xi_M} = -\mathscr{L}_{\xi_M} \tilde{J}_M$ defined by ξ_M.

Theorem 4.1. If for each $X \in \mathscr{G}$ the vector field $X^{H'} \in \tilde{\mathscr{G}}$, then ξ_M is locally submersive. Furthermore, if $\tilde{\mathscr{G}}$ is regular, then ξ_M is submersive.

Proof. In fact, we apply Theorem 3.1 to the involutive tangent distribution $E = \tilde{\mathscr{G}}$. It only remains to prove that $\tilde{\mathscr{G}}$ is Γ_{ξ_M}- and $\mathscr{L}_{\xi_M} \Gamma_{\xi_M}$-invariant. To prove this, let us remark that $\{X^V, X^{H'}\}$ spans $\tilde{\mathscr{G}}$. Then, from $\Gamma_{\xi_M}(X^V) = -X^V$, $\Gamma_{\xi_M}(X^{H'}) = X^{H'}$ we deduce that $\tilde{\mathscr{G}}$ is Γ_{ξ_M}-invariant.

Now, since

$$[\xi_M, X^V] = 2X^{H'} - X^{C_0}$$

we deduce that $[\xi_M, X^V] \in \tilde{\mathscr{G}}$ for any $X \in \mathscr{G}$. Then

$$(\mathscr{L}_{\xi_M} \Gamma_{\xi_M})(X^V) = -(\mathrm{Id} + \Gamma_{\xi_M})[\xi_M, X^V] \in \tilde{\mathscr{G}}$$

Also, since $X \in \mathscr{G}$ is an autonomous Lie symmetry, then we have

$$(\mathscr{L}_{\xi_M} \Gamma_{\xi_M})(X^{H'}) = [\xi_M, X^{H'}] - \Gamma_{\xi_M}[\xi_M, X^{H'}] = (\mathrm{Id} - \Gamma_{\xi_M})[\xi_M, X^{H'}]$$

But

$$X^{H'} = X^{C_0} - \frac{1}{2}([\xi_M, X^V] + X^{C_0})$$

implies

$$(\mathscr{L}_{\xi_M} \Gamma_{\xi_M})(X^{H'}) = -\frac{1}{2}(\mathrm{Id} - \Gamma_{\xi_M})[\xi_M[\xi_M, X^V] + X^{C_0}]$$

because of $[\xi_M, X^{C_0}] = 0$. Since $[\xi_M, X^V] + X^{C_0} \in \tilde{\mathscr{G}}$ and it is vertical we deduce that

$$(\mathscr{L}_{\xi_M}\Gamma_{\xi_M})(X^{H'}) \in \tilde{\mathscr{G}}$$

Consequently, ξ_M is locally submersive. Finally, if, moreover, $\tilde{\mathscr{G}}$ is J_M-regular, then the result follows from Theorem 3.1. ■

Proposition 4.1. Suppose that $\mathscr{G}$ is an Abelian Lie algebra of autonomous Lie symmetries of dimension $m - n$ such that $X^{H'} = X^{C_0}$ for any $X \in \mathscr{G}$. Then ξ_M is locally submersive and there exists a local coordinate system (x^α, y^a), $1 \leq \alpha \leq n$, $1 \leq a \leq m - n$, around each point of M such that (4) may be written as follows:

$$\frac{d^2x^\alpha}{dt^2} = \xi^\alpha_M\left(t, x^b, \frac{dx^b}{dt}\right)$$

$$\frac{d^2y^a}{dt^2} = \xi^a_M\left(t, x^b, \frac{dx^b}{dt}\right)$$

Proof. Since $X^{H'} = X^{C_0}$, for any $X \in \mathscr{G}$, then $X^{H'} \in \tilde{\mathscr{G}}$. Thus, from Theorem 4.1 we deduce that ξ_M is locally submersive. On the other hand, since $\mathscr{G}$ is Abelian, we can choose local coordinates (x^α, y^a) around each point of M such that

$$\mathscr{G} = \left\langle \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{m-n}} \right\rangle$$

Now, from

$$\left(\frac{\partial}{\partial y^a}\right)^{H'} = \left(\frac{\partial}{\partial y^a}\right)^{C_0} = \frac{\partial}{\partial y^a}$$

we obtain

$$\frac{\partial \xi^\beta_M}{\partial v^a} = \frac{\partial \xi^b_M}{\partial v^a} = 0, \qquad 1 \leq a, b \leq m - n, \qquad 1 \leq \beta \leq n$$

Also,

$$0 = \left[\left(\frac{\partial}{\partial y^a}\right)^{C_0}, \xi_M\right] = \left[\frac{\partial}{\partial y^a}, \xi_M\right]$$

implies

$$\frac{\partial \xi^\beta_M}{\partial y^a} = \frac{\partial \xi^b_M}{\partial y^a} = 0 \qquad \forall b, \beta$$

Hence we have

$$\xi_M^\alpha = \xi_M^\alpha\left(t, x^b, \frac{dx^b}{dt}\right), \qquad \xi_M^a = \xi_M^a\left(t, x^b, \frac{dx^b}{dt}\right)$$

which implies the required result. ■

Let $L\colon \mathbb{R} \times TM \to \mathbb{R}$ be a regular Lagrangian with Euler–Lagrange vector field ξ_L. We say that a vector field X on $\mathbb{R} \times M$ is a *symmetry* of L if $X^{(1)}L = 0$. We only consider symmetries of L which are vector fields on M, which will be called *autonomous symmetries* of L. Thus, a vector field X on M is an autonomous symmetry of L if and only if $X^{(1)}L = X^{C_0}L = 0$. This terminology is justified by the following fact. Let Φ_t be the flow on M generated by X. Then $\mathrm{Id}_{\mathbb{R}} \times T\Phi_t$ is the flow generated by X^{C_0}. Hence, if X is an autonomous symmetry of L then we deduce that Ω_L and dt are $\mathrm{Id}_{\mathbb{R}} \times T\Phi_t$-invariant. Consequently ξ_L is $\mathrm{Id}_{\mathbb{R}} \times T\Phi_t$-invariant, too, so that $[X^{C_0}, \xi_L] = 0$, i.e., X is an autonomous Lie symmetry of ξ_L.

Moreover, we have the following result.

Theorem 4.2. Let X be an autonomous symmetry of L and set $\mathscr{G} = \langle X \rangle$. Then (i) $X^V L$ is a first integral of L, and (ii) ξ_L is locally submersive if and only if

$$X^{H'} = X^{C_0} + \lambda X^V$$

where $\lambda\colon \mathbb{R} \times TM \to \mathbb{R}$.

Proof. (i) Since $i_{\xi_L}\Omega_L = 0$, $i_{\xi_L} dt = 1$, we deduce

$$\begin{aligned}
0 = (i_{\xi_L}\Omega_L)(X^{C_0}) &= -dd_{J_M}L(\xi_L, X^{C_0}) + (dE_L \wedge dt)(\xi_L, X^{C_0}) \\
&= -\xi_L(X^V L) + X^{C_0}(C_M L) + (J_M[\xi_L, X^{C_0}])L - X^{C_0}E_L \\
&= -\xi_L(X^V L) + X^{C_0}L
\end{aligned}$$

since $J_M[\xi_L, X^{C_0}] = 0$ and $E_L = C_M L - L$. Then $X^{C_0}L = 0$ implies $\xi_L(X^V L) = 0$.

(ii) We set $\mathscr{G} = \langle X \rangle$. Suppose that $X^{H'} = X^{C_0} + \lambda X^V$ for some function $\lambda\colon \mathbb{R} \times TM \to \mathbb{R}$. Then $X^{H'} \in \tilde{\mathscr{G}}$. Hence, from Theorem 4.1 it follows that ξ_M is locally submersive.

Conversely, suppose that ξ_L is locally submersive. We know that

$$X^{H'} = X^{C_0} - \frac{1}{2}([\xi_L, X^V] + X^{C_0})$$

and $Z = -\frac{1}{2}([\xi_L, X^V] + X^{C_0})$ is vertical. Then, if $\tilde{\mathscr{G}}$ is Γ_L-invariant we have

$$\Gamma_L X^{C_0} = -[\xi_L, X^V] + \tilde{J}_M[\xi_L, X^{C_0}] = -[\xi_L, X^V]$$

which implies $[\xi_L, X^V] \in \tilde{\mathcal{G}}$. Then $X^{H'} \in \tilde{\mathcal{G}}$ and consequently $Z \in \tilde{\mathcal{G}}$. Thus, we have $Z = \lambda X^V$ for some function $\lambda\colon \mathbb{R} \times TM \to \mathbb{R}$. ■

5. AN EXAMPLE

Let $L\colon \mathbb{R} \times T\mathbb{R}^3 \to \mathbb{R}$ be a regular Lagrangian given by

$$L(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) = \dot{x}\dot{y} + \dot{y} + \frac{1}{2} e^z \dot{z}^2 - e^x f(t)$$

where $(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$ stands for the induced coordinates on $\mathbb{R} \times T\mathbb{R}^3$ and $f\colon \mathbb{R} \to \mathbb{R}$. Then we have

$$\alpha_L = \dot{y}\, dx + (\dot{x} + 1)\, dy + e^z \dot{z}\, dz - \left(\dot{x}\dot{y} + \frac{1}{2} e^z \dot{z}^2 + e^x f(t)\right) dt$$

$$\Omega_L = dx \wedge d\dot{y} + dy \wedge d\dot{x} + e^z\, dz \wedge d\dot{z} + \dot{x}\, d\dot{y} \wedge dt + \dot{y}\, d\dot{x} \wedge dt$$

$$+ \dot{z} e^z\, d\dot{z} \wedge dt + \frac{1}{2} e^z \dot{z}^2\, dz \wedge dt + e^x f(t)\, dx \wedge dt$$

and the Euler–Lagrange vector field ξ_L is given by

$$\xi_L = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} - e^x f(t)\frac{\partial}{\partial \dot{y}} - \frac{1}{2}\dot{z}^2\frac{\partial}{\partial \dot{z}}$$

Then the Euler–Lagrange equations are

$$\begin{aligned} \frac{dx}{dt} &= \dot{x}, & \ddot{x} &= 0 \\ \frac{dy}{dt} &= \dot{y}, & \ddot{y} &= -e^x f(t) \\ \frac{dz}{dt} &= \dot{z}, & \ddot{z} &= -\frac{1}{2}\dot{z}^2 \end{aligned} \tag{6}$$

We know that $\partial/\partial y$ is an autonomous symmetry of L and a direct computation shows that

$$\left(\frac{\partial}{\partial y}\right)^{C_0} = \left(\frac{\partial}{\partial y}\right)^{H'}$$

Hence, from Theorem 4.2 we deduce that ξ_L is globally submersive with $\lambda = 0$. Furthermore, the global submersion is given by

$$\rho\colon \mathbb{R}^3 \to \mathbb{R}^2, \qquad \rho(x, y, z) = (x, z)$$

and then the projected nonautonomous SODE on $\mathbb{R} + T\mathbb{R}^2 \cong \mathbb{R}^5$ is

$$\xi_{\mathbb{R}^2} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{z}\frac{\partial}{\partial z} - \frac{1}{2}\dot{z}^2\frac{\partial}{\partial \dot{z}}$$

Thus equations (6) become

$$\begin{aligned} \frac{dx}{dt} &= \dot{x}, \qquad \ddot{x} = 0 \\ \frac{dz}{dt} &= \dot{z}, \qquad \ddot{z} = -\frac{1}{2}\dot{z}^2 \end{aligned} \tag{7}$$

We remark that (6) are time-dependent, while (7) do not depend on the time. In fact, $\xi_{\mathbb{R}^2} - \partial/\partial t$ is a SODE on $T\mathbb{R}^2$.

Now, let us recall the cosymplectic reduction procedure introduced by Albert (1989) (see also Cantrijn *et al.*, 1992; de León and Saralegui, n.d.).

Suppose that there exists a left action $\Phi: G \times M \to M$ of a Lie group G on a cosymplectic manifold (M, Ω, η). We always assume that both G and M are connected. The Lie algebra of G will be denoted by $\mathscr{G}$ and its dual by $\mathscr{G}^*$. For each $g \in G$ we put $\Phi_g \equiv \Phi(g, \cdot)$, the induced transformation on M. The fundamental vector field associated with $A \in \mathscr{G}$ is the vector field A_M on M defined by

$$A_M(x) = \frac{d}{dt}\Phi(\exp tA, x)\big|_{t=0}$$

An action Φ of a Lie group G on a cosymplectic manifold (M, Ω, η) is called *cosymplectic* if for each $g \in G$ the corresponding Φ_g is an automorphism of the cosymplectic structure, i.e., $\Phi_g^*\Omega = \Omega$, $\Phi_g^*\eta = \eta$.

A momentum map is a function $J: M \to \mathscr{G}^*$ such that if we define

$$J_A(x) = \langle A, J(x)\rangle$$

for all $A \in \mathscr{G}$, then $R(J_A) = 0$ and the Hamiltonian vector field X_{J_A} is just A_M. The momentum map J is said to be *Ad*-equivariant* if

$$J \circ \Phi_g = \mathrm{Ad}^*_{g^{-1}} \circ J$$

for each $g \in G$, where Ad* is the co-adjoint representation of G on $\mathscr{G}^*$.

For given $\mu \in \mathscr{G}^*$ we denote by G_μ the isotropy group of μ. By the Ad*-equivariance of J it follows that $J^{-1}(\mu)$ in an invariant subset for the restriction of Φ to G_μ. Moreover, if μ is a regular value of J, then $J^{-1}(\mu)$ is a submanifold of M and Φ induces a smooth action of G_μ on $J^{-1}(\mu)$. Following Libermann and Marle (1987), we will say that this action is *simple* if the orbit space $J^{-1}(\mu)/G_\mu$ admits a manifold structure such that the canonical projection $\pi_\mu: J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ is a surjective submersion. This will be, for instance, the case if the action is free and proper. In the sequel it will always be assumed that G_μ is connected and such that the fibers of π_μ are also connected.

Albert (1989; see also Cantrijn *et al.*, 1992; de León and Saralegui, n.d.) has established the following cosymplectic reduction theorem.

Theorem 5.1. There exists a unique cosymplectic structure (Ω_μ, η_μ) on the quotient space $M_\mu = J^{-1}(\mu)/G_\mu$ such that

$$j_\mu^* \Omega = \pi_\mu^* \qquad \text{and} \qquad j_\mu^* \eta = \pi_\mu^* \eta_\mu$$

with $j_\mu : J^{-1}(\mu) \to M$ the inclusion map and $\pi_\mu : J^{-1}(\mu) \to M_\mu$ the canonical projection. Further, the restriction of the Reeb vector field R to $J^{-1}(\mu)$ projects onto M_μ and its projection R_μ is just the Reeb vector field for the reduced cosymplectic structure (Ω_μ, η_μ).

Now suppose that H is a Hamiltonian function on M such that it is G-invariant, i.e., $H \circ \Phi_g = H$, for any $g \in G$. Then $H \circ j_\mu$ projects onto a function H_μ defined on M_μ. Denote by X_H the evolution vector field determined by H. Then X_H is tangent to $J^{-1}(\mu)$ and it projects onto M_μ to a vector field $(X_H)_\mu$ which is precisely the evolution vector field X_{H_μ} determined by H_μ on the reduced cosymplectic manifold M_μ. Hence the dynamics on M is projected onto the dynamics on M_μ. Notice that

$$\dim M_\mu = \dim M - \dim G - \dim G_\mu$$

and thus we have reduced the number of motion equations. The problem now is to reconstruct the dynamics. This may be in general a difficult problem.

We next apply this reduction procedure to the Lagrangian L. The Lie group is $G = \mathbb{R}$ and the action

$$\Phi: \quad \mathbb{R} \times (\mathbb{R} \times T\mathbb{R}^3) \to \mathbb{R} \times T\mathbb{R}^3$$

is given by

$$\Phi(s, (t, x, y, z, \dot{x}, \dot{y}, \dot{z})) = (t, x, y + s, z, \dot{x}, \dot{y}, \dot{z})$$

In other words, $\Phi_s : \mathbb{R} \times T\mathbb{R}^3 \to \mathbb{R} \times T\mathbb{R}^3$ is just $\Phi_s = \mathrm{Id}_{\mathbb{R}} \times T\phi_s$, where $\phi_s : \mathbb{R}^3 \to \mathbb{R}^3$ is the flow generated by $\partial/\partial y$. A direct computation shows that the action is cosymplectic for the cosymplectic structure (Ω_L, dt). In fact, this result follows from the invariance of L under the action. Also, E_L is $\mathbb{R}$-invariant.

A momentum map for the action is given by

$$\langle J(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), A \rangle = \langle \alpha_L(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), A_{\mathbb{R} \times T\mathbb{R}^3} \rangle$$

$A \in \mathbb{R}$. Thus we obtain

$$J(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) = \dot{x} + 1$$

Since $0 \in \mathbb{R}$ is a regular value, we have a reduced cosymplectic structure $(\tilde{\Omega}_L, dt)$ on $J^{-1}(0)/G = \mathbb{R} \times \mathbb{R}^4$. Further, ξ_L is tangent to $J^{-1}(0)$ and $\mathbb{R}$-invariant and thus it projects onto a vector field $\tilde{\xi}_L$ given by

$$\tilde{\xi}_L = \frac{\partial}{\partial t} - \frac{\partial}{\partial x} + \dot{z}\frac{\partial}{\partial z} - e^{-x}f(t)\frac{\partial}{\partial \dot{y}} - \frac{1}{2}\dot{z}^2\frac{\partial}{\partial \dot{z}}$$

The integral curves of $\tilde{\xi}_L$ satisfy the following system of differential equations:

$$\begin{aligned}
&\frac{dx}{dt} = -1\\
&\frac{dz}{dt} = \dot{z}\\
&\frac{d\dot{y}}{dt} = -e^{-x}f(t)\\
&\frac{d\dot{z}}{dt} = \ddot{z} = -\frac{1}{2}\dot{z}^2
\end{aligned} \tag{8}$$

In both cases the reduced manifold is just $\mathbb{R}^5$, but the reduced vector field is different, say $\xi_{\mathbb{R}^2} \neq \tilde{\xi}_L$. Further, $\tilde{\xi}_L$ is not a nonautonomous SODE. Only the fourth equation is of order 2. This example gives an illustration of the differences between both procedures of reduction.

The above example fits in a more general situation which we shall briefly describe.

Let $L: \mathbb{R} \times TM \to \mathbb{R}$ be a nonautonomous regular Lagrangian and suppose that G is a Lie group acting on M in such a way that L is G-invariant, say

$$L \circ (\mathrm{Id}_{\mathbb{R}} \times T\Phi_g) = L$$

where $\Phi_g: M \to M$ is the transformation of M defined by $g \in G$. The Lie algebra of G will be denoted by $\mathscr{G}$ and its dual by $\mathscr{G}^*$. Then the fundamental vector field associated with $A \in \mathscr{G}$ is an autonomous symmetry of L. Thus, we have a Lie subalgebra of autonomous Lie symmetries of ξ_L and we can apply the results obtained in Section 4 in order to decide if ξ_L is submersive or not. In the affimative case, we can reduce the dynamics to obtain the solutions of the projected nonautonomous SODE ξ_N with respect to a submersion $\rho: M \to N$.

Alternatively, since L is G-invariant, then the action of G on the cosymplectic manifold $\mathbb{R} \times TM$ with cosymplectic structure (Ω_L, dt) is cosymplectic and then we can apply the cosymplectic reduction procedure. To do this, we define a map $J: \mathbb{R} \times TM \to \mathscr{G}^*$ by

$$\langle J(t, q^i, v^i), A \rangle = \langle \alpha_L(t, q^i, v^i), A_{\mathbb{R} \times TM} \rangle$$

$A \in \mathscr{G}$. Thus we obtain

$$J(t, q^i, v^i) = J^i_a(q) \frac{\partial L}{\partial v_i} e^a$$

where $\{e^a, 1 \le a \le \dim G\}$ is a basis of $\mathscr{G}^*$. If $\partial J/\partial t = 0$, then J is a momentum map and we can apply Theorem 5.1. Now, the Hamiltonian function is just the energy E_L, which is in fact G-invariant. As the above example shows, if the Euler–Lagrange vector field ξ_L is submersive, we obtain finer information.

6. OUTLOOK

A natural question is to extend the results of this paper to a more general situation. In fact, we can consider a fibration $\pi_M: M \to S$, where S is a one-dimensional manifold. Then we define a SODE as a section $\xi: J^1\pi_M \to J^2\pi_M$ of the fibration $(\pi_M)^2_1: J^2\pi_M \to J^1\pi_M$, where $J^r\pi_M$ denotes the manifold of r-jets of sections of π_M (Saunders, 1989; Vondra, 1990). The problem may be set as follows: when are there two submersions $\rho: M \to N$ and $\pi_N: N \to S$ such that $\pi_N \circ \rho = \pi_M$ and $T\rho^2_1(\xi)$ is a SODE on $J^1\pi_N$? In this paper we have considered the case of trivial fibrations $M = S \times M'$, where $S = \mathbb{R}$.

We shall study in a forthcoming paper the characterization of submersive autonomous differential equations of higher order.

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